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On Subsystem Recognition in Compound Physical Systems†

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After a discussion on property lattices, I introduce a category of such lattices and structure-preserving maps. A careful analysis of the notions of physical system and of subsystem leads to the construction of a particular family of morphisms that are used for the construction of the coproduct in this category.

1. INTRODUCTION

In the approach to the foundations of physics that was initiated by the so-called Geneva School (Aerts, 1982; Jauch, 1968; Moore, 1999; Piron 1976, 1990), a theory is constructed of which the basic concepts and postulates are, at least partially, operationally motivated. A *particular physical system* is conceived as some part of the external phenomenal world that can be conceptually separated from its surroundings in the sense that its interaction with the environment can either be ignored or described in a simple way (Moore, 1999). It is clear that the notion of physical system depends more or less on the point of view of the physicist, hence is susceptible to a certain degree of arbitrariness and idealization.

Following Aerts (1982), I shall start with a well-defined class of *primitive questions* $\mathcal{P}$ one can perform on a given physical system and which is as complete as possible. Each primitive question α corresponds to one real experimental procedure one can perform on the physical system, and the performance of the experiment should yield two possible results: either it confirms the result the physicist anticipated or it gives a negative result. A

† This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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simple interchange of the positive and negative result gives the inverse question $\alpha^{\sim} \in \mathcal{P}$. Next, one introduces the notions of product question $\Pi_i \alpha_i$, a preorder on the obtained set of questions $\mathcal{Q}$, and one then proceeds with the construction of a complete lattice of properties $\mathcal{L} = \mathcal{Q}/\approx = \{[\alpha] | \alpha \in \mathcal{Q}\}$. It is a basic assumption that the possible states of a physical system are in a one-to-one correspondence with the set of atoms of the lattice of properties. Also, two states p and q are said to be *orthogonal* if there exists a question $\alpha \in \mathcal{Q}$ such that α is true if the system is in state p and $\alpha^{\sim}$ is true if it is in state q. It is not difficult to prove that $p \perp q \Leftrightarrow \exists \alpha \in \mathcal{P}$: $p \leq [\alpha]$ & $q \leq [\alpha^{\sim}]$. This orthogonality relation can be extended to $\mathcal{L} \backslash \{0\}$ by defining $a \perp b \Leftrightarrow \forall p \leq a, \forall q \leq b$: $p \perp q$. With slight abuse of language, one also puts $0 \perp a, \forall a \in \mathcal{L}$. To obtain a quantum-like theory with superselection rules, Piron (1976) and Aerts (1982) added several postulates to the lattice of properties associated with the physical system.

We shall use the notation $\mathcal{L}_p$ for the set of properties generated by the primitive questions $\mathcal{P}$. Because $\mathcal{L}_p$ generates the lattice by taking arbitrary meets, it is an order generating set in the opposite lattice $\mathcal{L}^{\text{OP}}$. Therefore we shall say that $\mathcal{L}_p$ is a *coorder generating set*. I start with the following definition.

Definition 1. A property lattice $\mathcal{L}$ is a complete atomistic lattice, with an order generating set of states and an orthocomplemented coorder generating set $\mathcal{L}_p$. The states Σ are the atoms of the property lattice.

It is an assumption that the orthocomplementation on $\mathcal{L}_p$ is the mapping $[\alpha] \mapsto [\alpha^{\sim}]$ for $\alpha \in \mathcal{P}$ (Aerts and Valckenborgh, n.d.). We shall also write $[\alpha^{\sim}] = [\alpha]^{\perp}$.

In contemporary mathematical thinking, it is customary to consider not only algebraic systems, but also mappings that conserve (some of) the relevant structure of these systems. Because the physical meaning of $\wedge_i a_i$ is operational conjunction and hence intrinsic to the family $\{a_i\}$, obvious candidates for such structure-preserving maps are maps $f: \mathcal{L} \rightarrow \mathcal{L}'$ that preserve arbitrary meets. One can also assume that $f(\mathcal{P}_p) \subseteq \mathcal{L}'_p$ such that $\forall a \in \mathcal{L}_p$: $f(a^{\perp}) = f(a)^{\perp}$. In this way, it is easy to see that we obtain a category that we shall call $\underline{\perp\text{MCLat}}$. For example, one can prove that in $\underline{\perp\text{MCLat}}$, the decomposition of a property lattice in components remains valid (Aerts and Valckenborgh, n.d.). Notice that this category is definitely *not* equal to the category *Prop* that was introduced by Moore (1995) in his categorical reformulation of the basic mathematical structures appearing in the Geneva approach.

2. THE RECOGNITION OF SUBSYSTEMS

In this paper, we shall restrict ourselves to the case of *nonidentical systems*. Suppose that we have two physical systems $\mathcal{S}$ and $\mathcal{S}'$ described by property lattices $\mathcal{L}$ and $\mathcal{L}'$. If the first entity can both occur as an individual, isolated physical system and as part of the second physical system, we expect that for every nontrivial question α relative to the first system, there exists an "equivalent" question α' that can be performed on the second, compound system. Therefore we have the induction of a map $f: \mathcal{L} \to \mathcal{L}': [\alpha] \mapsto [\alpha']$. If $\mathcal{L}$ is to be a subsystem of $\mathcal{L}'$, we expect that the map f should preserve the structure of the first physical system. Also implicit in the notion of a subsystem is the idea that one cannot invent new experiments to perform on the subsystem itself.

Definition 2. A subsystem map is a one-to-one map $f: \mathcal{L} \to \mathcal{L}': [\alpha] \mapsto [\alpha']$ subject to the following conditions:

(1) $f(\wedge_i a_i) = \wedge_i f(a_i)$
(2) $f(0) = 0'$
(3) $\forall c' \in \mathcal{L}'_P: [a \neq 0 \;\&\; f(a) \leq c'] \Rightarrow \exists\, c \in \mathcal{L}_P: f(c) = c'$
(4) $\forall \alpha \in \mathcal{P}: \exists \alpha' \in \mathcal{P}': \alpha' \in f([\alpha]) \;\&\; f([\alpha])^\perp = f([\alpha]^\perp)$

These conditions seem to be a *sine qua non* to recognize subsystems in physical systems on the propositional level. Note that every subsystem map is a morphism in $\perp$*MCLat*.

Lemma 1. $\forall\, a, b, a_i \in \mathcal{L}, \forall p, q \in \Sigma$: (1) $\forall c' \in \mathcal{L}': [a \neq 0 \;\&\; f(a) \leq c'] \Rightarrow \exists c \in \mathcal{L}: f(c) = c'$; (2) $f(1) = 1'$; (3) $f(\vee_i a_i) = \vee_i f(a_i)$; (4) $p \perp q \Rightarrow f(p) \perp f(q)$.

Proof: (1) Let $c' \in \mathcal{L}'$ and $0' \neq f(a) \leq c'$, with $c' = [\Pi_j \gamma'_j]$, $\gamma'_j \in \mathcal{P}'$, $\forall j$; hence, $\forall j: f(a) \leq [\gamma'_j] \Rightarrow \forall j: \exists \gamma_j \in \mathcal{P}: [\gamma'_j] = f([\gamma_j]) \Rightarrow c' = \wedge_j f([\gamma_j]) = f([\Pi_j \gamma_j])$; (2) $\forall\, a \in \mathcal{L}\backslash\{0\}: f(a) \leq 1' \Rightarrow \exists\, c \in \mathcal{L}: f(c) = 1' \Rightarrow a \leq c \Rightarrow c = 1$; (3) Let $\vee_i a_i \neq 0$ and $\forall i: f(a_i) \leq c'$ for $c' \in \mathcal{L}'$; then $\exists\, c \in \mathcal{L}: c' = f(c)$; hence $\forall i: a_i \leq c \Leftrightarrow \vee_i a_i \leq c \Leftrightarrow f(\vee_i a_i) \leq c'$; (4) $\exists \alpha \in \mathcal{P}: p \leq [\alpha], q \leq [\alpha]^\perp \Rightarrow f(p) \leq f([\alpha]), f(q) \leq f([\alpha]^\perp) = f([\alpha])^\perp$.

What can we say on the level of the state space descriptions Σ and Σ' of the two physical systems $\mathcal{S}$ and $\mathcal{S}'$? In the restricted case where it is legitimate to consider the subsystem as sufficiently isolated from the rest of the system, hence as a physical system in our sense, if $\mathcal{S}'$ is in a pure state $p' \in \Sigma'$, the state of the subsystem should be also determined as $p \in \Sigma$, i.e., we can define a map $p' \mapsto p$. Hence, we can construct a map $g: \Sigma'\backslash E' \to \Sigma: p' \mapsto p$, where E' are the states of the compound system for which the subsystem is not sufficiently isolated. One can remark that it should be possible in principle to extend the map g to the whole of Σ', but in that case one should allow for the description of the subsystem in terms of new quantities one might call 'non-pure states.' For example, it is conceivable that the experimental isolation of $\mathcal{S}$ from the larger $\mathcal{S}'$ can be described in terms of statistical quantities. Assume now that $\mathcal{S}$ is sufficiently isolated from the rest of $\mathcal{S}'$. If $\mathcal{S}'$ is in a state p' such that $f(a)$ is actual, then it is obvious that the measurement of $\alpha \in a$ on $\mathcal{S}$ should be independent of $\mathcal{S}$ being considered as a subsystem or not:

$$\forall p' \in \Sigma' \backslash E': \quad p' \leq f(a) \Leftrightarrow g(p') \leq a \tag{1}$$

If we note that Equation (1) resembles a Galois connection, we can construct the maps g: $\Sigma' \backslash E' \to \Sigma$.[3] The following result is well known. A proof can be found, for example, in Moore (1995).

Lemma 2. If f is $\wedge$-preserving, then the condition $\forall a, b$: $\hat{g}(a') \leq a \Leftrightarrow a' \leq f(a)$ has a unique solution $\hat{g}(a') = \wedge\{a | a' \leq f(a)\}$. $\hat{g}$ is $\vee$-preserving.

Proposition 3. From (1), g can be uniquely constructed from f. Moreover, we can take $\Sigma' \backslash E' = \{p' \in \Sigma | \exists p \in \Sigma: p' \leq f(p)\}$; g is surjective and $\forall$ $p', q' \in \Sigma \backslash E'$: $g(p') \perp g(q') \Rightarrow p' \perp q'$.

Proof: First we prove uniqueness: If g_1: $\Sigma' \backslash E' \to \Sigma$, g_2: $\Sigma' \backslash E' \to \Sigma$ satisfy the conditions, then $\forall$ $p' \in \Sigma' \backslash E'$: $g_1(p') \leq g_1(p') \Leftrightarrow p' \leq f \circ g_1(p') \Leftrightarrow g_2(p') \leq g_1(p')$. Hence g is the restriction of $\hat{g}$. We prove that $\{p' \in \Sigma' | \exists\, p \in \Sigma: p' \leq f(p)\}$ is the maximal possible domain: $p' \in \Sigma' \backslash E' \Leftrightarrow \hat{g}(p') \in \Sigma \Rightarrow \exists p \in \Sigma$: $\hat{g}(p') \leq p \Leftrightarrow \exists p \in \Sigma$: $p' \leq f(p)$. Conversely, if $\exists$ $p \in \Sigma$: $p' \leq f(p)$, then $\hat{g}\,(p') \leq p$. Now $\hat{g}(p') = 0 \Rightarrow \forall$ $a \in \mathcal{L}$: $\hat{g}(p') \leq a \Leftrightarrow \forall$ $a \in \mathcal{L}$: $a' \leq f(a) \Rightarrow p' = 0$. Hence, $\hat{g}(p') = p \in \Sigma$. Let $p \in \Sigma$; take $\Sigma' \ni p' \leq f(p)$; then $p' \in \Sigma' \backslash E' \Rightarrow 0 < g(p') \leq p \Rightarrow g(p') = p$. If $\exists$ $\alpha \in \mathcal{P}$: $g(p') \leq [\alpha]$, $g(q') \leq [\alpha^{\sim}] \Rightarrow p' \leq f([\alpha])$, $q' \leq f([\alpha^{\sim}])$; because $f([\alpha]) \in \mathcal{L}'_p$, the assertion follows. ■

From a physical point of view, it is no surprise that the domain of the map g consists exactly of those states of the compound system such that the subsystem itself is maximally specified.

3. THE COPRODUCT OF PROPERTY LATTICES

At this point, we have constructed property lattices, as mathematical representatives for the description of physical systems, and two kinds of mappings between property lattices: on one hand, a family of mappings that are general structure-preserving morphisms, on the other hand, mappings that express our subsystem idea. In the case that there is no interaction between the subsystems, a question that can be performed on one subsystem does not

[3] A Galois connection consists of a pair $\hat{g} \dashv f$ of order-preserving maps f: $\mathcal{L} \rightharpoonup \mathcal{L}'$ and $\hat{g}$: $\mathcal{L}' \to \mathcal{L}$, defined on preordered classes $\mathcal{L}$ and $\mathcal{L}'$ such that $\forall$ a, a': $\hat{g}(a') \leq a \Leftrightarrow a' \leq f(a)$.

have any influence on the other subsystems, and every subsystem can be considered as a genuine physical system in our sense.

In category theory, a *sink* is a pair consisting of an object A and a set-indexed family of morphisms g_i: $A_i \to A$. One says that a sink $(A_i \xrightarrow{c_i} C)_{i \in I}$ C is a *coproduct* if for every sink $(A_i \xrightarrow{f_i} A)_{i \in I}$ there exists a *unique* morphism $C \xrightarrow{f} A$ such that $f \circ c_i = f_i$, $\forall_i \in I$. It is well known and easy to see that products and coproducts, if they exist, are essentially unique, that is, unique up to isomorphism (Adámek *et al.*, 1990). Hence, it makes sense from a mathematical and physical point of view to look for the existence of a coproduct in $\perp MCLat$. Aerts (1984) already proved the existence of the coproduct of two property lattices $\mathscr{L}_1 \amalg \mathscr{L}_2$ in a related category.[4] I shall extend his construction and prove in this way the existence of a coproduct in general.

Theorem 4. If $\{\mathscr{L}_i\}_{i \in I}$ is a (set-indexed) family of property lattices, then $(\mathscr{L}_i \xrightarrow{c_i} \amalg_{i \in I} \mathscr{L})_{i \in I}$ is the coproduct in the category $\perp MCLat$ of property lattices, with

$$\coprod_{i \in I} \mathscr{L}_i = \{ f \,|\, f: I \to \cup_{i \in I} \mathscr{L}_i,\ f(i) \in \mathscr{L}_i \backslash \{0_i\}\ \forall i \in I \} \cup \{0\}$$

$$\forall f, g \in \coprod_{i \in I} \mathscr{L}_i: \quad f \le g \Leftrightarrow f(i) \le_i g(i) \quad \forall i \in I, \quad 0 \le f$$

$$\wedge \{ f_j | j \in J \} = \begin{cases} f: \quad f(i) = \wedge_{j \in J} f_j(i) \quad \forall i \in I & \text{if} \quad \forall\ i \in I: f(i) \neq 0_i \\ 0 & \text{if} \quad \exists i \in I: f(i) = 0_i \end{cases}$$

$$\vee \{ f_j | j \in J \} = f: \quad f(i) = \vee_{j \in J} f_j(i) \qquad \forall i \in I$$

and an order generating set of states $\Pi_{i \in I} \Sigma_i$ and subsystem maps:

$$c_i: \quad \mathscr{L}_i \to \coprod_{i \in I} \mathscr{L}_i: \quad a_i \mapsto c_i(a_i) \neq \begin{cases} f_{a_i} & \text{if} \quad a_i \neq 0_i \\ 0 & \text{if} \quad a_i = 0_i \end{cases}$$

where $J \neq \emptyset$ and $f_{a_i}(i) = a_i$, $f_{a_i}(j) = 1_j$ if $j \neq i$.

I give the proof in a series of lemmas. If convenient I shall write $a(i) = a_i$.

Lemma 5. $\amalg_{i \in I} \mathscr{L}_i$ is a complete atomic lattice with an order generating set of atoms $\Pi_{i \in I} \Sigma_i$ and an orthocomplemented coorder generating set $\mathscr{L}_P = \cup_i c_i (\mathscr{L}_{P_i})$.

Proof. We only prove the last assertion. It is not difficult to see that the set $\cup_i c_i (\mathscr{L}_{P_i})$ generates the lattice $\amalg_{i \in I} \mathscr{L}_i$ by taking arbitrary meets. More-

[4] For one or another reason, he called it the tensor product of two property lattices.

over, since each $\mathcal{L}_{P_i}$ is an orthocomplemented poset, also $\cup_i c_i(\mathcal{L}_{P_i})$ is: if $f_{a_i} \in c_i(\mathcal{L}_{P_i})$, then $f_{a_i^\perp}$ is the orthocomplement of f_{a_i}: $(f_{a_i})^\perp = f_{a_i^\perp}$. ■

Lemma 6. The mappings $c_i: \mathcal{L}_i \to \amalg_{i \in I} \mathcal{L}_i$ are subsystem maps.

Proof. (1) $c_i(\wedge_j a_i^j) = f_{\wedge_j a_i^j} = \wedge_j f_{a_i^j} = \wedge_j c_i(a_i^j)$; (3) let $d \in \mathcal{L}_p$, then $\exists i$: $d \in c_i(\mathcal{L}_{P_i}) \Rightarrow \exists b_i \in \mathcal{L}_{P_i}$: $d = f_{b_i}$; (4) let $a \in \mathcal{L}_P$; then $\exists i$: $a = c_i(a_i)$ and $c_i(a_i^\perp) = f_{a_i^\perp} = (f_{a_i})^\perp$. ■

Lemma 7. $\forall p, q \in \Pi_{i \in I} \Sigma_i$: $p \perp q \Leftrightarrow \exists i \in I{:}p(i) \perp q(i)$

Proof. If $p(j) \perp q(j)$, then $\exists\, a_j \in \mathcal{L}_{P_j}$ such that $p_j \le a_j$, $q_j \le a_j^\perp$, hence $c_j(p_j) \le c_j(a)_j$, $c_j(q)_j \le c_j(a_j^\perp) = c_j(a_j)^\perp \Rightarrow c_j(p_j) \perp c_j(q_j)$. Since $p \le c_j(p(j))$, $q \le c_j(q(j)) \Rightarrow p \perp q$. Conversely, if $p \perp q$, $\exists\, a \in \mathcal{L}_P$, $p \le a$, $q \le a^\perp$. Since $\exists i$: $a \in c_i(\mathcal{L}_{P_i}) \Rightarrow \exists a_i \in \mathcal{L}_{P_i}$: $a = f_{a_i} \Rightarrow p(i) \le a_i$, $q(\mathrm{i}) \le a_i^\perp$. ■

Lemma 8. $\forall\, a, b \in \amalg_{i \in I} \mathcal{L}_i$: $a \perp b \Leftrightarrow \exists k \in I$: $a(k) \perp b(k)$

Proof. Assume that $\exists k \in I$: $a(k) \perp b(k)$. Let $p \le a$, $q \le b \Leftrightarrow \forall_i \in I$: $p(i) \le a(i)$, $q(i) \le b(i) \Rightarrow p(k) \perp q(k) \Leftrightarrow p \perp q \Rightarrow a \perp b$. Conversely, assume that $\forall_i \in I$: $a_i \not\perp b_i \Rightarrow \forall_i \in I$: $\exists\, p_i \le a_i$: $\exists\, q_i \le b_i$: $p_i \not\perp q_i$. Construct p, q: $p(i) = p_i\ \forall i \in I$; then $p \le a$, $q \le b$ and $p \not\perp q$. Therefore $a \not\perp b$. ■

Lemma 9. $(\mathcal{L}_i \xrightarrow{c_i} \amalg_{i \in I} \mathcal{L})_{i \in I}$, is the coproduct in $\perp$*MCLat*.

Proof. For any sink $(\mathcal{L}_i \xrightarrow{d_i} \mathcal{L}^*)_{i \in I}$, we have to prove that there exists a unique morphism d: $\amalg_{i \in I} \mathcal{L}_i \to \mathcal{L}^*$ such that $\forall_i \in I$: $d_i = d \circ c_i$. Let $0 \neq a \in \amalg_{i \in I} \mathcal{L}_i$; then $\forall i$: $a(i) \neq 0_i$. We construct a map d: $\amalg_{i \in I} \mathcal{L}_i \to \mathcal{L}^*$: $a \mapsto \wedge_i d_i\,(a_i)$ for $a \neq 0$ and $d(0) = 0^*$. It is easy to see that d is $\wedge$-preserving and $d \circ c_j\,(a_j) = d(f_{aj}) = \wedge_{i \in I, i \neq j} d_i\,(1_i) \wedge d_j\,(a_j) = d_j(a_j)$ and $d \circ c_j(0_j) = d(0) = 0^* = d_j(0_j)$. If $a \in \mathcal{L}_P$, then $\exists a_i \in \mathcal{L}_{Pi}$: $f_{a_i} = a$. Hence $d(a^\perp) = d_i(a_i^\perp) = d_i(a_i)^\perp = d(a)^\perp$. Finally, for an arbitrary morphism $\hat{d}$: $\amalg_{i \in I} \mathcal{L}_i \to \mathcal{L}^*$ that satisfies the requirements, we have, since for $a \neq 0$: $a = \wedge_i f_{a_i}$, $\hat{d}(a) = \wedge_i \hat{d}(f_{a_i}) = \wedge_i \hat{d} \circ c(a_i) = \wedge_i d_i(a_i) = d(a)$, and this proves the uniqueness. ■

By its definition, the coproduct of a family of property lattices is the minimal compositional ingredient of the property lattices associated with the isolated subsystems into a compound system, in the sense that there has to exist a structure-preserving map from the coproduct to every other composition of the property lattices. The coproduct in $\perp$*MCLat* is weakly modular, but neither orthocomplemented nor does it satisfy the covering law (Aerts and Valckenborgh, n.d.).

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